Coefficient of restitution for viscoelastic spheres: The effect of delayed recovery

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The coefficient of normal restitution of colliding viscoelastic spheres is computed as a function of the material properties and the impact velocity. From simple arguments it becomes clear that, in a collision of purely repulsively interacting particles, the particles lose contact slightly before the distance of the centers of the spheres reaches the sum of the radii, that is, the particles recover their shape only after they lose contact with their collision partner. This effect was neglected in earlier calculations, which leads erroneously to attractive forces and thus to an underestimation of the coefficient of restitution. As a result we find a different dependence of the coefficient of restitution on the impact rate.

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I. INTRODUCTION

The dynamics of a granular system is governed by the particle interaction law, that is, by the forces the particles in contact exert on one another. In general, these forces may be complicated functions of the time-dependent mutual deformation and relative velocities in the normal and tangential directions. In the simplest case the particles are modeled as spheres interacting via normal and tangential forces.

Given particles of radii $R_{1/2}$ and masses $m_{1/2}$ at positions $\vec{r}_1(t)$ and $\vec{r}_2(t)$ traveling at velocities $\vec{v}_1(t)$ and $\vec{v}_2(t)$, the particle deformation is then described by

$$\xi(t) \equiv \max(0, R_1 + R_2 - |\vec{r}_1 - \vec{r}_2|) \tag{1}$$

and the deformation rate $\dot{\xi}(t)$. Apart from material properties, the dissipative and elastic components of the normal force of particles in contact depend on the deformation, the deformation rate, and the radii,

$$F = F^{(\mathrm{el})}(\xi, R_1, R_2) + F^{(\mathrm{dis})}(\xi, \dot{\xi}, R_1, R_2). \tag{2}$$

The functional form of these forces is model specific; see, e.g., [1–3]. Having specified the interaction forces, the dynamics of a ensemble of granular particles can be solved by a (force-based) molecular dynamics scheme.

An alternative approach uses the concept of the coefficient of restitution, relating the normal component of a pair of particles before and after a collision,

$$\varepsilon \equiv -\dot{\xi}(t_c)/\dot{\xi}(0). \tag{3}$$

This concept does not consider the duration of a contact, that is, a collision is an instantaneous event. Consequently, it is assumed that the particles collide exclusively pairwise. This condition is justified if the mean flight time between collisions is much larger than the duration of a collision, which restricts the range of applicability of the coefficient of restitution. The material properties of the particles are thus assumed to assure short duration of contact and/or that the particle number density of the system is small enough (low collision frequency) to neglect multiparticle contacts. In practical applications, event-driven molecular dynamics simulations, based on the coefficient of restitution, deliver frequently satisfying results even for rather dense systems.

Both concepts, interaction forces and the coefficient of restitution, can be applied to describe the dynamics of a granular system using either force-based or event-driven molecular dynamics. Describing the same physical systems, of course, the coefficient of restitution and the interaction forces must be closely related. Indeed, integration of Newton's equation of motion for an isolated pair of particles colliding at time $t=0$,

$$m_{\mathrm{eff}}\ddot{\xi} + F(\dot{\xi}, \xi) = 0, \quad \xi(0) = 0, \quad \dot{\xi}(0) = v, \tag{4}$$

with $m_{\mathrm{eff}} \equiv m_1 m_2/(m_1 + m_2)$ and

$$v \equiv \frac{[\vec{v}_1(0) - \vec{v}_2(0)] \cdot [\vec{r}_1(0) - \vec{r}_2(0)]}{R_1 + R_2}, \tag{5}$$

to obtain the trajectory $\xi(t)$, the coefficient of restitution is

$$\varepsilon = -\dot{\xi}(t_c)/v, \tag{6}$$

where t_c is the duration of the collision. This computation has been performed for several interaction force models [1,2,4,5]. Albeit conceptually simple, even for simple force laws the algebra is rather technical.

It is important that Eq. (2) applies to *particles in contact*. Obviously, in the absence of adhesion, the interaction force between colliding granular particles is strictly repulsive. Formally, however, during the decompression phase where $\dot{\xi} < 0$ the dissipative term $F^{(\mathrm{dis})}$ in Eq. (2) may overcompensate the pure repulsive conservative force $F^{(\mathrm{el})}$, erroneously yielding an attractive total force (e.g., [6,7]).

In molecular dynamics simulations, therefore, the normal force between particles is usually computed as $F^* = \max(0, F)$, with F given in Eq. (2) which assures that only repulsive forces act. The force F^* can thus be conveniently used in simulations.

The described artifact of a negative interaction force originates from an inappropriate definition of the end of a collision at time t_c. The duration of the collision, t_c, however, is needed for the derivation of the coefficient of restitution by means of Eq. (6). Whereas the beginning of a contact is well described by the condition $\xi(0)=0$, the end of a collision at time t_c is less trivial.

For simplicity of the computation in the literature it was assumed that the end of a collision is determined by the condition

$$\xi(t_c) = 0 \quad \text{with } t_c > 0. \tag{7}$$

As described above, in the decompression phase it may happen that $F(\xi,\dot{\xi})<0$. This means the collision may be completed even before $\xi=0$. Thus, the surfaces of the particles lose contact slightly before the distance of their centers exceeds the sum of their radii. Consequently, the deformation of the particles may last longer than the time of contact, and the particles gradually recover their spherical shape *after* they lose contact. The definition of the end of a collision,

$$F(t_c) = 0 \quad \text{with } t_c > 0, \tag{8}$$

takes the described scenario into account and assures that the particles interact exclusively repulsively.

Obviously, since erroneous attractive forces are excluded by the improved condition for the end of collision, the resulting coefficient of restitution is expected to be larger for the definition Eq. (8) than the value obtained for the condition Eq. (7).

Let us demonstrate the influence of the definition of t_c to the coefficient of restitution for the simplest form of the interaction force, the linear dash pot,

$$F(\xi,\dot{\xi}) = k\xi + \gamma\dot{\xi}. \tag{9}$$

Although neither the elastic nor the dissipative components are appropriate for the description of dissipatively colliding spheres (see below), the linear dash-pot model is frequently used in molecular dynamics simulations of granular systems. The main advantage of this model is the impact-velocity-independent coefficient of restitution, which follows from Eq. (6). Using the condition (7), we obtain for the case of low damping (e.g., [1,2])

$$\varepsilon = \exp\left(-\frac{\beta\pi}{\omega}\right) \quad \text{and} \quad t_c = \frac{\pi}{\omega} \tag{10}$$

with $\omega \equiv \sqrt{\omega_0^2 - \beta^2}$, $\omega_0 \equiv \sqrt{k/m_{\text{eff}}}$, and $\beta \equiv \gamma/2m_{\text{eff}}$. The above result holds only for $\beta < \omega_0$, for larger values of β we have $\varepsilon=0$. Obviously, this result contradicts the assumption of nonattractive interaction since

$$F(t_c) = F(\xi(t_c),\dot{\xi}(t_c)) = F(0,-\varepsilon v) = -\gamma\varepsilon v < 0. \tag{11}$$

For the condition Eq. (8) for t_c, taking into account that there is only repulsive interaction between granular particles, we find [8]

$$\varepsilon_n = \begin{cases} \exp\left[-\dfrac{\beta}{\omega_n}\left(\pi - \arctan\dfrac{2\beta\omega_n}{\omega_n^2-\beta^2}\right)\right], & \beta < \dfrac{\omega_0}{\sqrt{2}}, \\ \exp\left(-\dfrac{\beta}{\omega_n}\arctan\dfrac{2\beta\omega_n}{\omega_n^2-\beta^2}\right), & \beta \in \left(\dfrac{\omega_0}{\sqrt{2}},\omega_0\right), \\ \exp\left(-\dfrac{\beta}{\omega_n}\ln\dfrac{\beta+\omega_n}{\beta-\omega_n}\right), & \beta > \omega_0. \end{cases} \tag{12}$$

It can be shown that the solutions Eqs. (10) and (12) are fundamentally different: for values of the parameter β/ω_0 above 1 the duration of the collision t_c diverges in the case of Eq. (10), that is, $\varepsilon=0$. Thus, the particles collide with finite velocity and stick together (dissipative capture), despite our precondition of purely nonattractive interaction. The solution Eq. (12) does not reveal this unphysical behavior. For a detailed discussion see [8].

The linear dash-pot model serves here only as an example to show that even for the simplest force laws the adequate characterization of the end of the collision modifies the known results for the coefficient of restitution in a nontrivial way. For the case of the linear dash pot, the definition of t_c, Eq. (7) or (8), changes the coefficient of restitution as a function of the material parameters k and γ; however, ε is independent of the impact velocity v in both cases.

It is the aim of this paper to compute the coefficient of restitution for the simplest physically consistent force law for viscoelastic spheres with regard to the definition Eq. (8) for t_c. We will see that the appropriate choice of the condition for the end of the collision not only changes the dependence of the coefficient of restitution on the material parameters but also the functional form of its dependence on the impact velocity.

As our main result, presented in Eqs. (41) and (42), we will show that for the definition of t_c due to Eq. (8) the coefficient of restitution ε is given by a series in powers of $v^{1/10}$ whereas for the definition of t_c according to Eq. (7), ε is a series in powers of $v^{1/5}$ [4]. To obtain this result in Secs. II and III we introduce the relevant interaction force and formulate the equation of motion in scaled variables. Section IV describes the trajectory of the particles, first disregarding the specific boundary conditions. The details of the calculation are explained in Appendix A. Sections V and VI lead to the main result, the coefficient of restitution, by incorporating the boundary conditions, Eq. (7) (naive) and Eq. (8) (including delayed recovery), respectively. The details of the calculations are in Appendixes B and C. Finally, in Sec. VII we discuss briefly the time-dependent motion of the particles during the collision.

II. VISCOELASTIC SPHERES

We write the interaction force law for viscoelastic spheres [9] as

$$F(\xi,\dot{\xi}) = -\rho\xi^{3/2} - \frac{3}{2}A\rho\sqrt{\xi}\dot{\xi}. \tag{13}$$

The elastic part is given by the Hertz contact force [10] with the elastic constant

$$\rho \equiv \frac{2Y\sqrt{R_{\text{eff}}}}{3(1-\nu^2)}, \tag{14}$$

where Y is the Young modulus, ν is the Poisson ratio, and the effective radius of the colliding pair $R_{\text{eff}} \equiv R_i R_j/(R_i+R_j)$. The dissipative part, $\sim\sqrt{\xi}\dot{\xi}$, was derived independently in [9,11,12] using different methods but only the method in [9] allows us to derive the dissipative constant

$$A \equiv \frac{1}{3}\frac{(3\eta_2-\eta_1)^2}{3\eta_2+2\eta_1}\left(\frac{(1-\nu^2)(1-2\nu)}{Y\nu^2}\right) \tag{15}$$

as a function of viscous material constants $\eta_{1/2}$ that relate the dissipative stress tensor to the deformation rate tensor [13] and the elastic constants Y and ν.

While the coefficient of restitution for the linear dash-pot model depends only on the material constants, it may be shown already from a dimensional analysis that for viscoelastic particles the coefficient of restitution cannot be independent of the impact velocity [5,14–17]. It may be shown, moreover, either by scaling arguments [5] or in a more accurate way by a rather technical analysis [4] that the coefficient of restitution depends on the impact velocity as $\varepsilon=\varepsilon(v^{1/5})$. The coefficient of restitution was obtained in [4] for the definition (7) as a series expansion in powers of $v^{1/5}$. (For an equivalent derivation for viscoelastic disks, see [18].) In the following we derive the coefficient of restitution for the end of the collision given by Eq. (8).

III. EQUATION OF MOTION

Newton's equation of motion for the collision of viscoelastic spheres reads

$$\ddot{\xi} + k\xi^{3/2} + \gamma\sqrt{\xi}\dot{\xi} = 0 \tag{16}$$

with initial conditions

$$\xi(0) = 0, \quad \dot{\xi} = v. \tag{17}$$

and the constants

$$k \equiv \frac{\rho}{m_{\text{eff}}}, \quad \gamma = \frac{3}{2}\frac{\rho A}{m_{\text{eff}}}. \tag{18}$$

The equation of motion contains three parameters k, γ, and v and two scales of time and length, which are not fixed so far. Thus, with a proper choice of scales the number of free parameters can be reduced to 1. As it turns out that the analytical computation cannot be carried out with the most natural choice of scale, we will discuss the choice of both scales in detail.

The natural unit of time is $t_{\text{scale}}=k^{-2/5}v^{-1/5}$, which is proportional to the duration of the undamped collision, and the natural unit of length is $\xi_{\text{scale}}=k^{-2/5}v^{4/5}$, which is proportional to the maximal deformation. Adopting both natural units would reduce the number of free parameters to 1, $\gamma k^{-3/5}v^{1/5}$ [5]. This indicates that the coefficient of restitution is a function of $v^{1/5}$. The corresponding equation of motion reads

$$\ddot{x}_{\text{nat}} + x_{\text{nat}}^{3/2} + \gamma k^{-3/5}v^{1/5}\dot{x}_{\text{nat}}\sqrt{x_{\text{nat}}} = 0, \tag{19}$$

$$x_{\text{nat}}(0) = 0, \tag{20}$$

$$\dot{x}_{\text{nat}}(0) = 1, \tag{21}$$

with x_{nat} being the deformation in the natural length scale and overdots indicating derivation with respect to the time (in natural scales). This natural scaling is indeed used for the numerical integration of the equation of motion. For the analytical computation presented here it is, however, not suitable. It turns out that it is not possible to derive a series expansion that is accurate for the whole course of the collision. Instead we will have to use a series expansion of the direct collision for the first part of the collision and a series expansion of the time-inversed collision for the second part. This precludes the use of the *velocity-dependent* natural scaling as both branches of the trajectory would be scaled with different length scales (the different time scales will not be a problem as we will merge both series expansions at the point in time where $\dot{\xi}=0$).

Instead of the natural scales discussed above we adopt the *velocity-independent* length scale $\xi_{\text{scale}}=k^{-2/5}$. We thus scale time and length as

$$\xi = \frac{x}{k^{2/5}}, \quad t = \frac{\tau}{k^{2/5}v^{1/5}}, \tag{22}$$

and arrive at the equation

$$\ddot{x} + \beta v^{-1/5}\dot{x}\sqrt{x} + v^{-2/5}x^{3/2} = 0, \tag{23}$$

$$x(0) = 0, \quad \dot{x}(0) = v^{4/5},$$

where overdots mean derivatives with respect to the scaled time τ and $\beta \equiv \gamma k^{-3/5}$. Note that the deformation ξ or x is counted positive if the particles deform each other. The impact velocity $\dot{\xi}(0)$ or $\dot{x}(0)$ has to be positive as its action *increases* the deformation.

IV. TRAJECTORY

First we have to determine the trajectory of the particles during the collision. To this end we apply the method that was introduced in [4].

First we observe that the trajectory cannot be a series in integer powers of time due to the fact that the third and higher time derivatives of the deformation are singular at $x=0$. The deformation $x=0$ corresponds to the start of the collision and also to its end under the condition Eq. (7). [Here we consider the collision for the condition Eq. (8); nevertheless, for the calculation we refer in several places to the end of the collision due to Eq. (7), which we call the *naive* end of the collision.] As an example for such a divergence, the third time derivative of x reads

$$x^{(3)} = -\frac{3}{2v^{2/5}}\dot{x}\sqrt{x} + \frac{\beta}{v^{3/5}}x^2 + \frac{\beta^2}{v^{2/5}}x\dot{x} - \frac{\beta}{2v^{1/5}}\frac{\dot{x}^2}{\sqrt{x}}. \tag{24}$$

The last term diverges for $x\to 0$ as for the beginning and the end of collision $\dot{x}\neq 0$. It turns out that instead of integer powers the trajectory is a series of half-integer powers of τ. The computation of the trajectory $x(\tau)$ is explained in detail in Appendix A. The first few terms read

$$x = v^{4/5}\Bigg[\tau - \frac{4}{15}\beta v^{1/5}\tau^{5/2} - \frac{4}{35}\tau^{7/2} + \frac{1}{15}\beta^2 v^{2/5}\tau^4 + \frac{3}{70}\beta v^{1/5}\tau^5 - \frac{38}{2475}\beta^3 v^{3/5}\tau^{11/2} + \frac{1}{175}\tau^6 - \frac{937}{75\,075}\beta^2 v^{2/5}\tau^{13/2} + \frac{2612}{779\,625}\beta^4 v^{4/5}\tau^7 - \frac{713}{238\,875}\beta v^{1/5}\tau^{15/2} + \frac{43\,943}{13\,513\,500}\beta^3 v^{3/5}\tau^8 - \left(\frac{22}{104\,125} + \frac{31\,159}{44\,178\,750}\beta^5 v\right)\tau^{17/2} + \frac{871}{808\,500}\beta^2 v^{2/5}\tau^9 - \frac{192\,113}{242\,492\,250}\beta^4 v^{4/5}\tau^{19/2}\Bigg] + O(\tau^{10}). \quad (25)$$

It turns out that this series converges very slowly, which means that we need the series up to a high order (see below). The structure of this result becomes clear if we sort the terms in escalating powers of the damping parameter β. The trajectory then takes the form

$$x = v^{4/5}\left(\tau - \frac{4}{35}\tau^{7/2} + \frac{1}{175}\tau^6 - \frac{22}{104\,125}\tau^{17/2} + \cdots\right) + \beta v\left(-\frac{4}{15}\tau^{5/2} + \frac{3}{70}\tau^5 - \frac{713}{238\,875}\tau^{15/2} + \cdots\right) + \beta^2 v^{6/5}\left(\frac{1}{15}\tau^4 - \frac{937}{75\,075}\tau^{13/2} + \frac{871}{808\,500}\tau^9 + \cdots\right) + \beta^3 v^{7/5}\left(-\frac{38}{2475}\tau^{11/2} + \frac{43\,943}{13\,513\,500}\tau^8 + \cdots\right) + \beta^4 v^{8/5}\left(\frac{2612}{779\,625}\tau^7 - \frac{192\,113}{242\,492\,250}\tau^{19/2}\right) + \beta^5 v^{9/5}\left(-\frac{31\,159}{44\,178\,750}\tau^{17/2} + \cdots\right) + O(\tau^{10}). \quad (26)$$

The expressions in parentheses do not contain any parameter except for pure numbers. They are hence universal functions, which we shall call $x_i(\tau)$, where the index i gives the power of β it is associated with. Note furthermore that subsequent powers of τ in each function differ by $5/2$. The trajectory can be written compactly as

$$x(\tau) = v^{4/5}x_0(\tau) + \beta v x_1(\tau) + \beta^2 v^{6/5}x_2(\tau) + \cdots = v^{4/5}\sum_{k=0}^{\infty}(\beta v^{1/5})^k x_k(\tau). \quad (27)$$

The function x_0 is the trajectory of the undamped ($\beta=0$) collision. It is known [5] that it reaches its maximal compression at time

$$\tau^0_{\max} = \left(\frac{4}{5}\right)^{3/5}\frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{9}{10}\right)} \approx 1.609. \quad (28)$$

The total duration of the undamped collision is $\tau^0_c=2\tau^0_{\max}$ as the undamped trajectory is symmetrical with respect to the point of maximal compression.

We proceed by computing the time of maximal compression of the damped problem along with the value of maximal compression. We use the ansatz

$$\tau_{\max} = \tau^0_{\max} + \sum_{k=1}^{\infty} a_k\beta^k v^{k/5} \quad (29)$$

and solve for the coefficients a_k as explained in detail in Appendix B. The first coefficients a_k are listed in Table III there. The principal form of these and other similar expressions—power series in $\beta v^{1/5}$—can be derived by scaling arguments detailed in [5]. The maximal deformation can be obtained by Taylor expansion of Eq. (26),

$$x_{\max} = v^{4/5}\sum_{k=0}^{\infty} b_k\beta^k v^{k/5}, \quad (30)$$

with the coefficients b_k. We will not need these coefficients explicitly; they can, nevertheless, be found in Table III.

V. FINAL VELOCITY FOR THE NAIVE CONDITION

Let us compute the final (naive) velocity, assuming the end of the collision according to Eq. (7). At first glance one might be tempted to compute the duration of collision with an ansatz like $\tau_c=\tau^0_c+\delta\tau_c$ and solve for the correction terms by performing a Taylor expansion around the undamped duration of collision. This method, however, fails due to the aforementioned singularity at $x=0$. Instead we compute the final velocity indirectly: as we have an expression that is definitely valid for the first part up to the maximal compression, we can construct the full solution by a kind of backward-shooting method. We start at the end of the collision where $\dot{x}=-v'$ (the final velocity v' being unknown yet) and let the time run backward. The equation of motion for this inverse collision is identical to Eq. (23),

$$\ddot{x}_{\mathrm{inv}} - \beta v'^{-1/5}\dot{x}_{\mathrm{inv}}\sqrt{x_{\mathrm{inv}}} + v'^{-2/5}x_{\mathrm{inv}}^{3/2} = 0,$$

$$x_{\mathrm{inv}}(0) = 0, \quad \dot{x}_{\mathrm{inv}}(0) = v'^{4/5}, \quad (31)$$

except for the sign of the damping parameter β, since the inverse collision (in inverse time) is an accelerated collision. Consequently, the trajectory of the inverse problem can be obtained from the solution of the direct collision, Eq. (25), by simply substituting $\beta\to-\beta$ and $v\to v'$,

$$x_{\rm inv} = v'^{4/5}\left(\tau + \frac{4}{15}\beta v'^{1/5}\tau^{5/2} - \frac{4}{35}\tau^{7/2} + \frac{1}{15}\beta^2 v'^{2/5}\tau^4 - \frac{3}{70}\beta v'^{1/5}\tau^5 + \frac{38}{2475}\beta^3 v'^{3/5}\tau^{11/2}\right) + O(\tau^6). \quad (32)$$

The same is true for the maximal compression of the inverse collision,

$$x_{\max}^{\rm inv} = v'^{4/5}\sum_{k=0}^{\infty}(-1)^k b_k \beta^k v'^{k/5}, \quad (33)$$

with the same numerical coefficients b_k as in Eq. (30).

As the inverse collision problem is just a reformulation of the original collision problem both maximal compressions have to be the same,

$$x_{\max} = x_{\max}^{\rm inv}, \quad (34)$$

which is an equation for v'. From these arguments the choice of our length scale, Eq. (22), becomes evident: if the natural unit of length $k^{-2/5}v^{-1/5}$ was chosen, the direct and the inverse collision problems would have different length scales as the initial velocity of the inverse collision is $v' \neq v$.

In order to solve Eq. (34) for v' we use the ansatz

$$v' = v + \beta\delta v_1 + \beta^2\delta v_2 + \cdots \quad (35)$$

and solve for the corrections δv_i. Using the definition Eq. (3) this yields a coefficient of restitution of the form

$$\varepsilon(v)^{\rm naive} = 1 + c_1\beta v^{1/5} + c_2\beta^2 v^{2/5} + \cdots . \quad (36)$$

Note that we determined the final velocity v' at $x=0$, that is, this result for $\varepsilon(v)$ corresponds to the condition Eq. (7) for the end of the collision. Based on the trajectory derived so far, in the next section we will derive the coefficient of restitution that corresponds to Eq. (8).

The calculation of the coefficients c_k in Eq. (36) is explained in Appendix B; the numerical values of the first coefficients are shown in Table III.

VI. PREMATURE END OF THE COLLISION

Up to here we have calculated the solution of the equation of motion, Eq. (16), in the interval from ($\xi=0$, $\dot{\xi}=v$) (start of the collision) to ($\xi=0$, $\dot{\xi}=v'$) (end of the collision) or the scaled equation (23) in the corresponding interval $x=0$ in the beginning and $x=0$ in the end, respectively. The velocity at the end of this trajectory, v', led us to the coefficient of restitution corresponding to the condition Eq. (7).

As discussed before, however, the velocity v' corresponds to a negative interaction force, in contradiction to the purely repulsive interaction of viscoelastic granular particles. Therefore, the collision does not end at $x=0$ but before, when the interaction force becomes zero. This condition corresponds to the condition Eq. (8).

We take this premature end of collision into account and thus look for the earliest point in time T during the inverse collision when the acceleration vanishes. Setting $\ddot{x}_{\rm inv}=0$ in Eq. (31) yields

$$\beta v'^{1/5}\dot{x}_{\rm inv}(T) = x_{\rm inv}(T). \quad (37)$$

For small $\beta v'^{1/5}$ we obtain T to lowest order by approximating $x_{\rm inv}$ by $v'^{4/5}\tau$ which yields

$$T \approx \beta v'^{1/5}. \quad (38)$$

The solution to higher order reads

$$T = \beta v'^{1/5} + \frac{4}{35}\beta^{7/2}v'^{7/10} + \frac{2}{75}\beta^6 v'^{6/5} + \frac{21\,271}{2\,734\,875}\beta^{17/2}v'^{17/10} + \cdots . \quad (39)$$

The details of this calculation can be reviewed in Appendix C. The value of $\dot{x}_{\rm inv}$ at this point in time is

$$\dot{x}_{\rm inv}(T) = v'^{4/5}\left(1 + \frac{4}{15}\beta^{5/2}v'^{1/2} + \frac{11}{210}\beta^5 v' + \cdots\right). \quad (40)$$

Going back to the original units of time we obtain the final velocity for the case of the condition Eq. (8),

$$v'' = \dot{\xi}_{\rm final} = v'\left(1 + \frac{4}{15}\beta^{5/2}v'^{1/2} + \frac{11}{210}\beta^5 v' + \cdots\right). \quad (41)$$

Inserting the expression for v', one arrives at the final solution

$$\varepsilon = 1 - 1.153\beta v^{1/5} + 0.798\beta^2 v^{2/5} + 0.267\beta^{5/2}v^{1/2} + \cdots = 1 + \sum_{k=0}^{\infty} h_k \beta^{k/2} v^{k/10}. \quad (42)$$

The details of this computation are shown in Appendix C. The coefficients h_k are pure numbers; the first 20 of them can be found (to a higher precision than in the expression above) in Table V. As the coefficient of restitution ε depends only on $\beta v^{1/5}$ (including half powers of this term) we show the velocity dependence in this universal form in Fig. 1.

The analytical results, Eqs. (36) and (42), are compared with the numerical solution of the equation of motion (23). In the interval shown in Fig. 1, the analytical results agree with the numerical results almost perfectly. Beyond the interval shown the solutions start to deviate. As an example, in physical units we consider a sphere that collides with $\varepsilon=0.8$ at $v=1$ m/s, e.g., a rubber sphere. By numerically solving Eq. (42) we obtain $\beta=0.2$ s$^{1/5}$/m$^{1/5}$. Consequently, the range of velocity shown in Fig. 1, $\beta v^{1/5} \leq 0.3$, corresponds to $v \leq 7.5$ m/s. From the good agreement between the analytical and numerical solutions in this interval we conclude that the range of validity of the solution, Eq. (42), is at least $v \leq 7.5$ m/s. For materials with smaller damping constant β the range of validity is larger.

Albeit in Fig. 1 numerical and analytical results almost coincide, we note that the deviation for the improved condition, Eq. (8), exceeds the deviation for the naive condition by several orders of magnitude. This can be seen from the coefficients h_k (see Table V), which decrease only slowly for increasing k. Thus, to obtain a good precision for $\beta v^{1/5}$ close to unity a very large number of coefficients h_k is required.

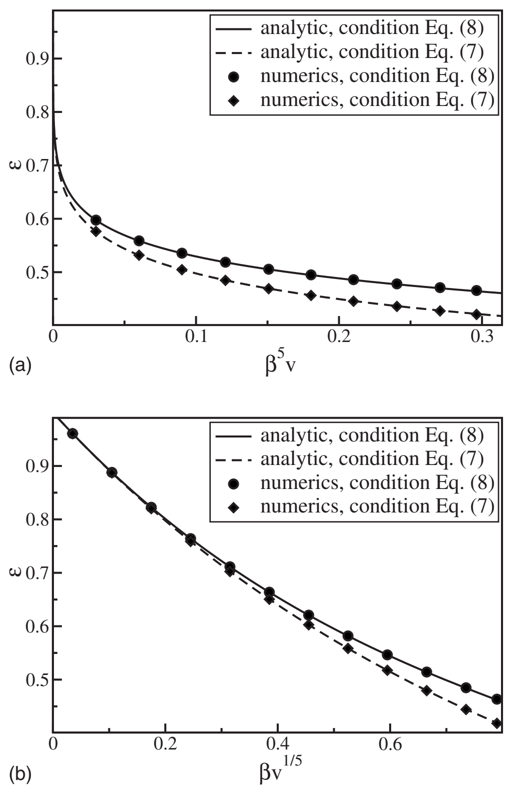

FIG. 1. Velocity dependence $\varepsilon(v)$ for both conditions for the end of the collision, Eq. (7) (naive) and Eq. (7) (improved). The upper panel shows the dependence on velocity, the lower panel shows the dependence on $\beta v^{1/5}$. For both panels the interval shown on the abscissa is (almost) equivalent. The numerical solution of Newton's equation of motion, Eq. (23), agrees almost perfectly with the analytical curves shown here. Beyond the shown interval there are increasing discrepancies between theory and simulation.

For large velocities or large damping both velocity dependencies, Eqs. (36) and (42), reveal a remarkable difference: For the naive condition, Eq. (7), the coefficient of restitution decays asymptotically as $\varepsilon \sim v^{-1}$. For the improved condition, Eq. (8), the asymptotics is compatible with a power law of $\varepsilon \sim v^{-0.331}$. Both asymptotics are shown in Fig. 2.

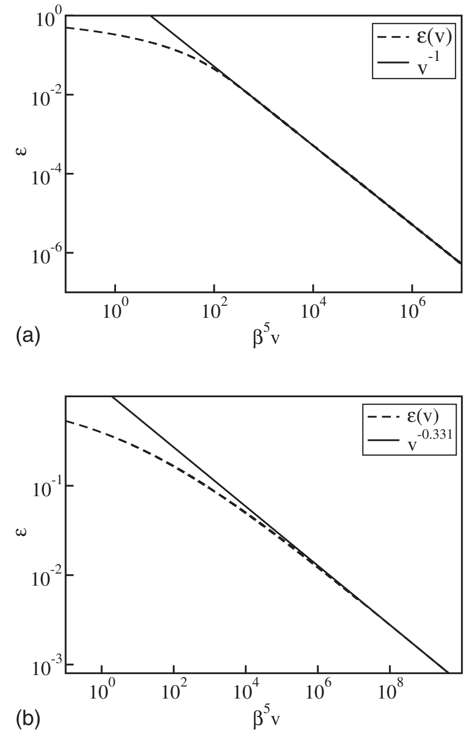

FIG. 2. Asymptotics for $\varepsilon(v)$ for the naive end-of-collision condition Eq. (7) (top) and the improved condition Eq. (8), (bottom). Both are representable by a simple power law. For the naive condition the exponent is -1, for the correct condition it is close to $-1/3$.

VII. DYNAMICS OF THE COLLISION

As the equation of motion is the same for both end-of-collision criteria, in both cases the particles follow the same trajectory. The only difference is the terminal moment. Figure 3 shows the scaled deformation $x(\tau)$, the scaled acceleration $\ddot{x}(\tau)$, and the scaled deformation rate $\dot{x}(\tau)$ as functions of the scaled time τ to illustrate the consequences of the choice of termination criterion. The end of the collision according to the criterion Eq. (8) is indicated by vertical dashed lines.

The main effect discussed in the paper is the resulting erroneous attractive force when the naive criterion, Eq. (7), is used, which can be seen in Fig. 3(b). After coming to a stop at the point of maximal compression the particles are *accelerated* away from each other. After the terminal point prescribed by the improved criterion, Eq. (8) (vertical dashed lines in Fig. 3) is reached, the attractive force may significantly *decelerate* the particles, yielding a much smaller final velocity [see Fig. 3(c)] (final value of the full line as compared to the value indicated by the horizontal dashed line). Thus, the naive criterion overestimates the damping and leads, consequently, to a too small value for the coefficient of restitution.

VIII. CONCLUSION

We described the collision of a pair of particles which interact repulsively according to the force law Eq. (13), valid for viscoelastic spheres. In a physically consistent description, the end of the collision is determined by the instant during the expansion when the interaction force vanishes, $\ddot{\xi}(t)=0$, (a) but not by the naive condition $\xi(t)=0$ (b), which corresponds to the instant when the distance of the centers of the particles coincides with the sum of their radii. This becomes obvious when the interaction force at the end of the collision is inspected: For condition (b) the interaction force becomes attractive, which contradicts the precondition of purely repulsive interaction. The reason for this behavior is the *delayed recovery* of the particles, that is, the surfaces of the particles already lose contact slightly before the compressed particles recovered their spherical shape.

The choice of the condition for the end of the collision, (a) or (b), has a drastic effect on the resulting velocity dependence $\varepsilon(v)$ of the coefficient of normal restitution. Instead of a series in $v^{1/5}$ obtained for the naive condition (b)

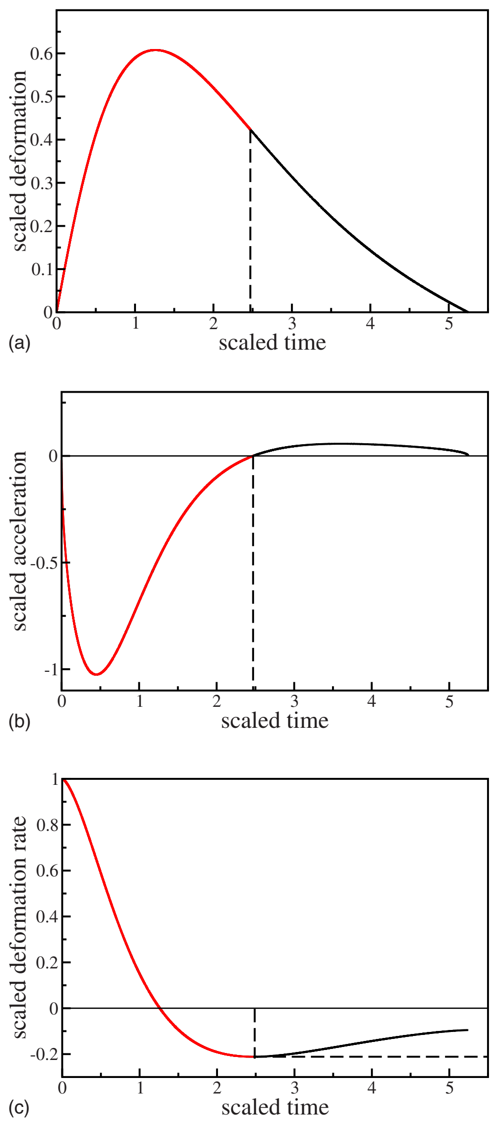

FIG. 3. (Color online) Comparison of the collision dynamics for both end-of-collsion criteria, Eqs. (7) and (8). When the improved criterion, Eq. (8), is used, the collision ends with positive scaled deformation $x(\tau_c)$ (a). After this point in time the interaction force $\ddot{x}(\tau)$ becomes attractive (b), yielding a much lower final velocity $\dot{x}(\tau_c)$ (c). Due to the high damping the final force for the naive criterion is small (but still positive).

[4,5], for the physically consistent end-of-collision condition (a) we obtain a series in $v^{1/10}$ where the odd powers of $v^{1/10}$ are solely due to the end-of-collision rule. The analytical results agree almost perfectly with the numerical integration of Newton's equation of motion for colliding viscoelastic spheres.

We evaluated the result for $\varepsilon(v)$ for realistic material properties for the cases (a) and (b) and obtained a noticeable difference of up to about 20%, depending on the material properties. The range of validity of our result was estimated to be about 10 m/s for a soft, rather dissipative material such as rubber. For a more elastic material, corresponding to a larger coefficient of restitution, the range of validity is significantly larger. Our analytical results deviate from the numerical results for $\beta v^{1/5} \gtrsim 0.9$, which may be attributed to the properties of the series Eq. (42), which converges slowly for large $\beta v^{1/5}$ and whose convergence is not even clear for $\beta v^{1/5} \geqslant 1$.

For large impact velocity we can, however, still obtain numerical results that reveal another drastic difference between the conditions (a) and (b). For both conditions, asymptotically $\varepsilon(v)$ follows a power law. For the naive condition (b), however, we obtain $\varepsilon \sim v^{-1}$, whereas for the physically consistent condition (a) we find $\varepsilon \sim v^{-1/3}$.

The influence of the end-of-collision condition on the coefficient of restitution for viscoelastic particles is in marked contrast to the corresponding result obtained for the linear dash-pot model [8]. Here the choice of the condition (a) or (b) would result in a modified coefficient of restitution which is, nevertheless, independent of the impact velocity in both cases.

ACKNOWLEDGMENT

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APPENDIX A: COMPUTATION OF THE TRAJECTORY

Equation (23) for the trajectory $x(\tau)$ of the particles' relative motion in the scaled variables x and τ,

$$\ddot{x} + \beta v^{-1/5}\dot{x}\sqrt{x} + v^{-2/5}x^{3/2} = 0$$

$$x(0) = 0, \quad \dot{x}(0) = v^{4/5}, \tag{A1}$$

is solved by series expansion. As explained in the text, an expansion in powers of τ fails; instead we expand in powers of $\sqrt{\tau}$. Using the ansatz

$$x(\tau) = v^{4/5}\tau[1 + R(\tau)], \tag{A2}$$

Eq. (A1) turns into

$$2\dot{R} + \tau\ddot{R} + \beta v^{1/5}\tau^{1/2}(1 + R + \tau\dot{R})\sqrt{1 + R} + \tau^{3/2}(1 + R)^{3/2} = 0,$$

$$R(0) = 0, \quad \dot{R}(0) = 0. \tag{A3}$$

The term R may be expanded in powers of $\sqrt{\tau}$,

$$R(\tau) = a_0 + a_1\tau^{1/2} + a_2\tau^1 + a_3\tau^{3/2}\cdots . \tag{A4}$$

Inserting Eq. (A4) into Eq. (A1) and comparing equal powers of $\tau^{1/2}$, we find $a_0=a_1=a_2=0$, that is, the first nontrivial contribution is $O(\tau^{3/2})$. This fact simplifies the subsequent computer algebra considerably.

We determine the coefficients $a_3, a_4, \ldots$ in escalating order using an iterative procedure. In the first step we determine a_3 while a_i $(i>3)$ stay undetermined. The corresponding term for R of the order 3 is denominated by $R_3 \equiv a_3\tau^{3/2}+O(\tau^2)$, the next order is 4 with $R_4 \equiv a_4\tau^2+O(\tau^{5/2})$, etc. In other words, R_i contains all contributions of order $O(\tau^{i/2})$ and higher. In each step i of the iteration we derive a differential equation $G_i[R_i, \dot{R}_i, \ddot{R}_i]=0$ for R_i.

We demonstrate the procedure for the first terms of a series up to the term $a_9\tau^{9/2}$. For the first step, $i=3$, we expand $(1+R)^{1/2}$ and $(1+R)^{3/2}$ in Eq. (A3) up to the necessary order for R_3. Since $N=9$ and the lowest order of τ in R is 3, we need the expansion up to the third term,

$$\sqrt{1+R_3} = 1 + \frac{R_3}{2} - \frac{R_3^2}{8} + \frac{R_3^3}{16},$$

$$(1+R_3)^{3/2} = 1 + \frac{3R_3}{2} + \frac{3R_3^2}{8} - \frac{R_3^3}{16}. \tag{A5}$$

Equation (A3) reads then

$$G_3[R_3] \equiv 2\dot{R}_3 + \tau\ddot{R}_3 + \beta v^{1/5}\tau^{1/2} \times\left(1 + \frac{3}{2}R_3 + \frac{3}{8}R_3^2 + \tau\dot{R}_3 + \frac{1}{2}\tau\dot{R}_3R_3\right) + \tau^{3/2}\left(1 + \frac{3}{2}R_3 + \frac{3}{8}R_3^2\right) = 0, \tag{A6}$$

where terms of order $\tau^{10/2}$ and higher are neglected.

The desired coefficient a_3 is now isolated by the formal transformation

$$R_3 = a_3\tau^{3/2} + R_4, \tag{A7}$$

which establishes the first iteration step. In general, we replace

$$R_i = a_i\tau^{i/2} + R_{i+1}, \tag{A8}$$

and insert this into $G_i(R_i)=0$ where only terms of relevant order are taken into account. Then we consider the term $O(\tau^{i/2-1})$ and determine a_i. After substituting a_i back into G_i, we are left with the next-order equation $G_{i+1}(R_{i+1})=0$, which is then solved in the same way, etc.

We insert Eq. (A7) into Eq. (A6) and obtain

$$2\dot{R}_4 + \tau\ddot{R}_4 + \left(\frac{15}{4}a_3 + \beta v^{1/5}\right)\tau^{1/2} + \tau^{3/2} + 3\beta v^{1/5}a_3\tau^2 + \frac{3}{2}a_3\tau^3 + \frac{9}{8}\beta v^{1/5}a_3^2\tau^{7/2} + \frac{3}{8}a_3^2\tau^{9/2} + \frac{3}{2}\beta v^{1/5}R_4\tau^{1/2} + \frac{3}{2}R_4\tau^{3/2} + \beta v^{1/5}\dot{R}_4\tau^{3/2} + \frac{3}{2}\beta v^{1/5}a_3R_4\tau^2 + \frac{3}{8}\beta v^{1/5}R_4^2\tau^{1/2} = 0, \tag{A9}$$

where again terms of irrelevant order were skipped. The term in parentheses of lowest order 1/2 ($i/2-1$ in general) allows for the computation of the first nontrivial coefficient $a_3=-(4/15)\beta v^{1/5}$. We insert a_3 into Eq. (A9) to obtain the next equation for the computation of a_4:

$$G_4[R_4] \equiv 2\dot{R}_4 + \tau\ddot{R}_4 + \frac{3}{2}\beta v^{1/5}R_4\tau^{1/2} + \frac{3}{8}\beta v^{1/5}R_4^2\tau^{1/2} + \tau^{3/2} + \frac{3}{2}R_4\tau^{3/2} + \beta v^{1/5}\dot{R}_4\tau^{3/2} - \frac{2}{5}\beta^2 v^{2/5}R_4\tau^2 - \frac{4}{5}\beta^2 v^{2/5}\tau^2 - \frac{2}{5}\beta v^{1/5}\tau^3 + \frac{2}{25}\beta^3 v^{3/5}\tau^{7/2} + \frac{2}{75}\beta^2 v^{2/5}\tau^{9/2} = 0. \tag{A10}$$

The next iteration step $R_4=a_4\tau^2+R_5$ leads to

$$2\dot{R}_5 + \tau\ddot{R}_5 + 6a_4\tau + \tau^{3/2} - \frac{4}{5}\beta^2 v^{2/5}\tau^2 + \frac{7}{2}\beta v^{1/5}a_4\tau^{5/2} - \frac{2}{5}\beta v^{1/5}\tau^3 + \left(\frac{3}{2}a_4 + \frac{2}{25}\beta^3 v^{3/5}\right)\tau^{7/2} + \frac{3}{2}\beta v^{1/5}R_5\tau^{1/2} + \frac{3}{2}R_5\tau^{3/2} + \beta v^{1/5}\dot{R}_5\tau^{3/2} - \frac{2}{5}\beta^2 v^{2/5}R_5\tau^2 - \frac{2}{5}\beta^2 v^{2/5}a_4\tau^4 + \frac{3}{8}\beta v^{1/5}a_4^2\tau^{9/2} + \frac{2}{75}\beta^2 v^{2/5}\tau^{9/2} = 0. \tag{A11}$$

From the terms of lowest order we find $0=6a_4\tau$, that is, $a_4=0$. We insert this into Eq. (A11):

$$G_5[R_5] \equiv 2\dot{R}_5 + \tau\ddot{R}_5 + \tau^{3/2} - \frac{4}{5}\beta^2 v^{2/5}\tau^2 - \frac{2}{5}\beta v^{1/5}\tau^3 + \frac{2}{25}\beta^3 v^{3/5}\tau^{7/2} + \frac{3}{2}\beta v^{1/5}R_5\tau^{1/2} + \frac{3}{2}R_5\tau^{3/2} + \beta v^{1/5}\dot{R}_5\tau^{3/2} - \frac{2}{5}\beta^2 v^{2/5}R_5\tau^2 + \frac{2}{75}\beta^2 v^{2/5}\tau^{9/2} = 0. \tag{A12}$$

With $R_5=a_5\tau^{5/2}+R_6$ the last equation turns into

$$2\dot{R}_6 + \tau\ddot{R}_6 + \left(\frac{35}{4}a_5 + 1\right)\tau^{3/2} - \frac{4}{5}\beta^2 v^{2/5}\tau^2 + \left(4\beta v^{1/5}a_5 - \frac{2}{5}\beta v^{1/5}\right)\tau^3 + \frac{2}{25}\beta^3 v^{3/5}\tau^{7/2} + \frac{3}{2}a_5\tau^4 + \left(-\frac{2}{5}\beta^2 v^{2/5}a_5 + \frac{2}{75}\beta^2 v^{2/5}\right)\tau^{9/2} + \frac{3}{2}\beta v^{1/5}R_6\tau^{1/2} + \beta v^{1/5}\dot{R}_6\tau^{3/2} + \frac{3}{2}R_6\tau^{3/2} = 0. \tag{A13}$$

From the lowest-order terms $O(\tau^{3/2})$ we obtain $a_5=-4/35$. We insert

$$G_6[R_6] \equiv 2\dot{R}_6 + \tau\ddot{R}_6 - \frac{4}{5}\beta^2 v^{2/5}\tau^2 - \frac{6}{7}\beta v^{1/5}\tau^3 + \frac{2}{25}\beta^3 v^{3/5}\tau^{7/2} - \frac{6}{35}\tau^4 + \frac{38}{525}\beta^2 v^{2/5}\tau^{9/2} + \frac{3}{2}\beta v^{1/5}R_6\tau^{1/2} + \frac{3}{2}R_6\tau^{3/2} + \beta v^{1/5}\dot{R}_6\tau^{3/2} = 0, \tag{A14}$$

iterate $R_6=a_6\tau^3+R_7$, and obtain

$$2\dot{R}_7+\ddot{R}_7\tau+\left(12a_6-\frac{4}{5}\beta^2 v^{2/5}\right)\tau^2-\frac{6}{7}\beta v^{1/5}\tau^3+\left(\frac{9}{2}\beta v^{1/5}a_6+\frac{2}{25}\beta^3 v^{3/5}\right)\tau^{7/2}-\frac{6}{35}\tau^4+\left(\frac{3}{2}a_6+\frac{38}{525}\beta^2 v^{2/5}\right)\tau^{9/2}+\frac{3}{2}\beta v^{1/5}R_7\tau^{1/2}=0. \quad (A15)$$

From the term of lowest order we find $a_6=\beta^2 v^{2/5}/15$. We insert a_6 into Eq. (A15) for the next-order equation,

$$G_7[R_7]\equiv 2\dot{R}_7+\tau\ddot{R}_7-\frac{6}{7}\beta v^{1/5}\tau^3+\frac{19}{50}\beta^3 v^{3/5}\tau^{7/2}-\frac{6}{35}\tau^4+\frac{181}{1050}\beta^2 v^{2/5}\tau^{9/2}+\frac{3}{2}\beta v^{1/5}R_7\tau^{1/2}=0. \quad (A16)$$

Iterating $R_7=a_7\tau^{7/2}+R_8$ yields

$$2\dot{R}_8+\ddot{R}_8\tau+\frac{63}{4}a_7\tau^{5/2}-\frac{6}{7}\beta v^{1/5}\tau^3+\frac{19}{50}\beta^3 v^{3/5}\tau^{7/2}+\frac{3}{2}\beta v^{1/5}a_7\tau^4-\frac{6}{35}\tau^4+\frac{181}{1050}\beta^2 v^{2/5}\tau^{9/2}+\frac{3}{2}\beta v^{1/5}R_8\tau^{1/2}=0 \quad (A17)$$

and from the lowest-order $\sim\tau^{5/2}$ we obtain $a_7=0$. We insert this solution, and substitute $R_8=a_8\tau^4+R_9$:

$$2\dot{R}_9+\tau\ddot{R}_9+\left(20a_8-\frac{6}{7}\beta v^{1/5}\right)\tau^3+\frac{19}{50}\beta^3 v^{3/5}\tau^{7/2}-\frac{6}{35}\tau^4+\left(\frac{3}{2}\beta v^{1/5}a_8+\frac{181}{1050}\beta^2 v^{2/5}\right)\tau^{9/2}=0. \quad (A18)$$

We insert the solution $a_8=(3/70)\beta v^{1/5}$:

$$G_9[R_9]\equiv 2\dot{R}_9+\tau\ddot{R}_9+\frac{19}{50}\beta^3 v^{3/5}\tau^{7/2}-\frac{6}{35}\tau^4+\frac{71}{300}\beta^2 v^{2/5}\tau^{9/2}=0, \quad (A19)$$

replace $R_9=a_9\tau^{9/2}+R_{10}$,

$$2\dot{R}_{10}+\ddot{R}_{10}\tau+\left(\frac{99}{4}a_9+\frac{19}{50}\beta^3 v^{3/5}\right)\tau^{7/2}-\frac{6}{35}\tau^4+\frac{71}{300}\beta^2 v^{2/5}\tau^{9/2}=0, \quad (A20)$$

and obtain $a_9=-(38/2475)\beta^3 v^{3/2}$. Inserting this solution and substituting $R_{10}=a_{10}\tau^5+R_{11}$ yields

$$2\dot{R}_{11}+\tau\ddot{R}_{11}+\left(30a_{10}-\frac{6}{35}\right)\tau^4+\frac{71}{300}\beta^2 v^{2/5}\tau^{9/2}=0 \quad (A21)$$

and thus $a_{10}=1/175$, which is the last coefficient that can be obtained from the expansion Eq. (A5).

TABLE I. MAPLE program [19] for the computation of the trajectory. This small piece of code is an elegant representation of the algebra discussed in Appendix A.

```
N:=150;
dgl:=2*Rd+s^2*Rdd+(A*s+s^3)*(1+R)^(3/2)
      +A*s^3*Rd*sqrt(1+R);
dgl:=convert(taylor(dgl,R,N),polynom):
solution:=0;
for i from 3 to N do
      dgl:=subs(Rdd=(i*(i-2)/4)*a*s^(i-4)+Rdd ,
       Rd=(i/2)*a*s^(i-2)+Rd,R=a*s^i+R,dgl):
      dgl:=mtaylor (dgl,[R,s,Rd,Rdd],N,[i,1,i,i]):
      tmp:=expand (coeff(coeff(coeff(dgl,R,0),
      Rdd,0),Rd,0));
      asol:=solve(coeff(tmp,s,i-2),a):
      print(i, asol):
      dgl:=simplify(subs(a=asol,dgl)):
      solution:=solution+asol*s^i:
end do:
solution:=v^(4/5)*s^2*(1+solution):
solution:=subs(A=beta*v^(1/5),solution):
fout:=fopen ("./solution,"WRITE);
fprintf(fout,"%a \n,"solution);
fclose(fout);
```

To achieve an acceptable accuracy of the final result, the expansion Eq. (42) has to be performed up to high orders in $\beta v^{1/5}$. To accurately compute the necessary coefficients h_k, one needs accurate functions x_k of the same index k. For the chosen accuracy (20 coefficients h_k) the expansion of the trajectory has to be performed up to an order as large as 150. We employ computer algebra (MAPLE) which turns the described algorithm into only a few lines of code (see Table I). For the computation we abbreviate $A\equiv\beta v^{1/5}$, $s\equiv\sqrt{\tau}$, Rd and Rdd stand for $dR/d\tau$ and $d^2R/d\tau^2$, and N is the order of the expansion.

APPENDIX B: TIME AND VALUE OF MAXIMAL COMPRESSION AND THE SERIES $\varepsilon_{\text{naive}}(v)$

The first ingredient for the actual computation of $\varepsilon_{\text{naive}}(v)$ is the maximum compression. To this end, we first compute at which time $\tau_{\max}$ this maximum compression is achieved. The time of maximum compression will be determined by Taylor expansion of the expression

$$\dot{x}(\tau_{\max}^0+\delta\tau)=0. \quad (B1)$$

Here the time $\tau_{\max}^0$ of maximum compression of the undamped collision as given by Eq. (28) is taken as a reference. In terms of the universal functions x_i, as introduced in Eq. (26), the Taylor expansion takes the form

TABLE II. MAPLE code [19] for the computation of the coefficients c_i in Appendix B.

```
restart;Digits:=20;
nb:=20;
fin:=fopen(“./solution”,READ):
xin:=fscanf (fin,“%a”):
x:=simplify(subs(s=sqrt(t),xin[1])):
tchalf:=(4/5)^(3/5)*GAMMA(2/5)*GAMMA(1/2)/
    (2*GAMMA(9/10)):
x:=subs(t=tchalf+dt,x):
xdot:=evalf(taylor(diff(x,dt),dt=0,nb)):
xdot:=convert(xdot,polynom):
dt:=sum(a[‘i’]*beta^‘i’*v^((1/5)*‘i’),‘i’=1..nb):
for i to nb do
    a[i]:=solve(coeff(xdot,beta,i),a[i])
od:
hh:=convert(evalf(taylor(x,beta,nb+1)),polynom):
xmax:=unapply(hh,v):
xmaxinv:=unapply(subs(beta=-beta,hh),v):
u:=v*(1+sum(c[‘k’]*beta^‘k’*v^((1/5)*‘k’) ,
     ‘k’=1..nb)):
d:=convert(taylor(xmax(v)-xmaxinv(u),beta,nb+1) ,
    polynom):
for i to nb do
    c[i]:=solve(coeff(d,beta,i),c[i])
od:
fout:=fopen(“./coefficients”,WRITE):
for i to nb do
    fprintf(fout,“%a\n”,c[i])
od:
fclose(fout):
```

$$\dot{x}(\tau_{\max}^0+\delta\tau)=v^{4/5}\sum_{i=0}^{n_\beta}\beta^i v^{i/5}\sum_{k=0}^{n_\beta}\frac{d^{k+1}x_i}{d\tau^{k+1}}\frac{\delta\tau^k}{k!},\qquad\text{(B2)}$$

which motivates the representation of $\delta\tau$ as a series of the form

$$\delta\tau=\sum_{n=1}^{n_\beta}a_n\beta^n v^{n/5}.\qquad\text{(B3)}$$

We insert Eq. (B3) into the Taylor expansion, collect coefficients in powers of β and solve successively for a_n. The result is shown in Table III.

In the same way the maximum compression $x_{\max}$ can be computed by performing the Taylor expansion of $x(\tau_{\max}^0+\delta\tau)$ which is of the form

$$x_{\max}=v^{4/5}\sum_{i=0}^{n_\beta}\beta^i v^{i/5}\sum_{k=0}^{n_\beta}\frac{d^k x_i}{d\tau^k}\frac{\delta\tau^k}{k!}.\qquad\text{(B4)}$$

suggesting the series

TABLE III. The first numerical coefficients of the expansions Eq. (B3).

i	a_i	b_i	c_i
0		1.093 362 074	
1	−0.286 747 122 0	−0.504 454 892 6	−1.153 448 854
2	0.104 858 992 2	0.284 043 019 2	0.798 266 555 3
3	−0.048 684 064 00	−0.170 220 777 6	−0.522 882 560 9
4	0.025 431 168 90	0.105 500 708 8	0.348 742 667 8
5	−0.014 236 582 82	−0.066 848 713 71	−0.233 098 126 0
6	0.008 337 660 013	0.043 039 492 29	0.156 682 147 7
7	−0.005 039 737 366	−0.028 051 084 30	−0.105 818 782 8
8	0.003 118 137 108	0.018 460 851 21	0.071 765 282 42
9	−0.001 964 027 745	−0.012 246 185 62	−0.048 857 172 37
10	0.001 254 701 962	0.008 177 589 114	0.033 373 471 94

$$x_{\max}=v^{4/5}\sum_{n=0}^{n_b}b_n\beta^n v^{n/5}.\qquad\text{(B5)}$$

The coefficients b_n are also shown in Table III. The first coefficient b_0 is the maximum compression for the undamped problem. To actually compute the coefficient of normal restitution without regard of the premature loss of contact we have to match the maximum compression of the direct and the inverse collision, i.e., we have to solve

$$x_{\max}(v)=x_{\max}^{\text{inv}}(v')\qquad\text{(B6)}$$

for v' with

$$x_{\max}(v)=v^{4/5}\sum_{n=0}^{n_\beta}b_n\beta^n v^{n/5},\qquad\text{(B7)}$$

$$x_{\max}(v')=v'^{4/5}\sum_{n=0}^{n_\beta}(-1)^n b_n\beta^n v'^{n/5}.\qquad\text{(B8)}$$

In the MAPLE program (Table II) the function $x_{\max}(v)$ is called h(v), the function $x_{\max}(v)$ is called hm(v). Using the ansatz

$$v'=v\left(1+\sum_{n=1}^{n_\beta}c_n\beta^n v^{n/5}\right)=v\,\varepsilon_{\text{naive}}(v),\qquad\text{(B9)}$$

we can solve for c_n by expanding the expression Eq. (B6) for small β and collect orders. The first c_k are shown in Table III.

APPENDIX C: PREMATURE LOSS OF CONTACT

As the moment of actual loss of contact is close to the naive end of contact we will use the inverse collision to compute the time T and velocity v'' at loss of contact. Using the condition $\ddot{x}_{\text{inv}}=0$, we obtain the equation for T:

$$\beta v'^{1/5}\dot{x}_{\text{inv}}(T)=x_{\text{inv}}(T).\qquad\text{(C1)}$$

Approximating x_{inv} as $v'^{4/5}T$, we obtain the leading order of

TABLE IV. MAPLE code [19] for the computation of the coefficients h_k due to Appendix C.

```
restart:Order:=20:nd:=4:Digits:=20:
fin:=fopen("./solution",READ):
L:=fscanf(fin,"%a"):fclose(fin):
x:=convert(taylor(L[1],s,Order),polynom):
xinv:=subs(s=sqrt(T),subs(beta=-B*B,x)):
xinvdot:=diff(xinv,T):
eqn:=simplify(xinv-B*B*v(1/5)*xinvdot):
T:=B^2*v^(1/5)*sum(d['k']*B^(5*'k')*v^('k'/2) ,
   'k'=0..nd):
eqn:=expand(eqn):
eqn:=series(eqn,B,2*Order+1):
for i from 0 to nd do
    d[i]:=solve(coeff(eqn,B,2+5*i),d[i]):
od:
vpp:=convert(series(v^(1/5)*xinvdot,B,2*Order+1) ,
    polynom):
fin:=fopen("./coefficients",READ):
for i from 1 to Order do
    L:=fscanf(fin,"%a"):
    c[i]:=L[1]:
od:
fclose(fin):
vprime:=v*(1+sum(c['k']*B^(2*'k')*v^('k'/5)  ,
     'k'=1..Order)):
vpp:=convert(series(subs(v=vprime,vpp),B,2*Order+1) ,
    polynom):
epsilon:=simplify(vpp/v);
fout:=fopen("./hk",WRITE):
for i from 1 to 2*Order do
    h[i]:=simplify(coeff(epsilon,B,i)/v^(i/10)):
    fprintf(fout,"%a,\n",h[i]):
od:
```

$$T = \beta v'^{1/5}. \tag{C2}$$

After canceling the common prefactor $v'^{4/5}$ Eq. (C1) depends only the combination $\beta v'^{1/5}$. Therefore, one can easily guess the principal form of T:

$$T = \beta v'^{1/5} + \sum_{k=2}^{\infty} d_k \beta^k v'^{k/5}. \tag{C3}$$

Inserting this ansatz into Eq. (C1), collecting orders, and solving for d_k yields

$$T = \beta v'^{1/5} + \frac{4}{35}\beta^{7/2}v'^{7/10} + \frac{2}{75}\beta^6 v'^{6/5} + \frac{21\,271}{2\,734\,875}\beta^{17/2}v'^{17/10} + \cdots . \tag{C4}$$

The final solution now reads

$$v'' = v\left(1 + \frac{4}{15}\beta^{5/2}v^{1/2} + \frac{13}{150}\beta^5 v + \frac{4897}{160\,875}\beta^{15/2}v^{3/2} + \frac{453\,263}{40\,540\,500}\beta^{10}v^2 + \cdots\right). \tag{C5}$$

Inserting the known solution for v', we obtain

$$v'' = v\left(1 + \sum_{k=0}^{\infty} h_k \beta^{k/2} v^{k/10}\right), \tag{C6}$$

$$\varepsilon(v) = 1 + \sum_{k=0}^{\infty} h_k \beta^{k/2} v^{k/10} \tag{C7}$$

The code used is given in Table IV and the first values of h_k in Table V.

TABLE V. The first numerical coefficients of the expansion (C6). The coefficients h_2 and h_4 are identical to the first coefficients in the original expansion $\varepsilon(v)$, i.e., $h_2 \equiv c_1$ and $h_4 \equiv c_2$.

i	h_i
0	1
1	0
2	−1.153 448 856
3	0
4	0.798 266 558 1
5	0.266 666 666 7
6	−0.522 882 565 7
7	−0.461 379 542 4
8	0.348 742 673 7
9	0.452 351 049 6
10	−0.146 431 464 4
11	−0.367 728 299 2
12	−0.043 248 983 3
13	0.281 804 232 5
14	0.147 852 587 2
15	−0.179 442 059 0
16	−0.178 466 032 6
17	0.065 933 588 82
18	0.171 358 617 8
19	0.025 249 822 3
20	−0.137 923 498 6

[1] J. Schäfer, S. Dippel, and D. E. Wolf, J. Phys. I **6**, 5 (1996).
[2] H. Kruggel-Emden, E. Simsek, S. Rickelt, S. Wirtz, and V. Scherer, Powder Technol. **171**, 157 (2007).
[3] T. Pöschel and T. Schwager, *Computational Granular Dynamics: Models and Algorithms* (Springer, Berlin, 2005).
[4] T. Schwager and T. Pöschel, Phys. Rev. E **57**, 650 (1998).
[5] R. Ramírez, T. Pöschel, N. V. Brilliantov, and T. Schwager, Phys. Rev. E **60**, 4465 (1999).
[6] S. Luding, in *Physics of Dry Granular Media*, edited by H. J. Herrmann, J.-P. J.-P. Hovi, and S. Luding (Kluwer Academic, Dordrecht, 1998), p. 285.
[7] S. Luding, *Die Physik kohäsionsloser granularer Medien* (Logos-Verlag, Berlin, 1998) (in German).
[8] T. Schwager and T. Pöschel, Granular Matter **9**, 465 (2007).
[9] N. V. Brilliantov, F. Spahn, J.-M. Hertzsch, and T. Pöschel, Phys. Rev. E **53**, 5382 (1996).
[10] H. Hertz, J. Reine Angew. Math. **92**, 156 (1882).
[11] G. Kuwabara and K. Kono, Jpn. J. Appl. Phys., Part 1 **26**, 1230 (1987).
[12] W. A. M. Morgado and I. Oppenheim, Phys. Rev. E **55**, 1940 (1997).
[13] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Oxford University Press, Oxford, 1965).
[14] T. Tanaka, T. Ishida, and Y. Tsuji, Trans. Jpn. Soc. Mech. Eng. BC **57**, 456 (1991).
[15] Y. Tsuji, T. Tanaka, and T. Ishida, Powder Technol. **71**, 239 (1991).
[16] Y.-h. Taguchi, J. Phys. I **2**, 2103 (1992).
[17] S. Luding, E. Clément, A. Blumen, J. Rajchenbach, and J. Duran, Phys. Rev. E **50**, 4113 (1994).
[18] T. Schwager, Phys. Rev. E **75**, 051305 (2007).
[19] See EPAPS Document No. E-PLEEE8-78-007811 for the MAPLE source code. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.